Effects of forcing in three dimensional turbulent flows

Luca Biferale[1,4], Alessandra S. Lanotte[2,4], and Federico Toschi[3,4]
[1] *Dip. di Fisica, Università "Tor Vergata", Via della Ricerca Scientifica 1, I-00133 Roma, Italy*
[2] *Istituto per le Scienze dell'Atmosfera e del Clima, CNR, Str. Prov. Lecce-Monteroni km. 1200, I-73100 Lecce, Italy*
[3] *Istituto per le Applicazioni del Calcolo, CNR, Viale del Policlinico 137, I-00161 Roma, Italy*
[4] *INFM, Unitá di Tor Vergata, I-00133 Roma, Italy*

We present the results of a numerical investigation of three-dimensional homogeneous and isotropic turbulence, stirred by a random forcing with a power law spectrum, $E_f(k) \sim k^{3-y}$. Numerical simulations are performed at different resolutions up to 512^3. We show that at varying the spectrum slope y, small-scale turbulent fluctuations change from a *forcing independent* to a *forcing dominated* statistics. We argue that the critical value separating the two behaviours, in three dimensions, is $y_c = 4$. When the statistics is forcing dominated, for $y < y_c$, we find dimensional scaling, i.e. intermittency is vanishingly small. On the other hand, for $y > y_c$, we find the same anomalous scaling measured in flows forced only at large scales. We connect these results with the issue of *universality* in turbulent flows.

The effects of both external forcing mechanisms and boundary conditions on small-scale turbulent fluctuations have been the subject of many theoretical, numerical and experimental studies [1, 2]. The 1941 theory of Kolmogorov [1] is based on the assumption of *local* isotropy and homogeneity, that is any turbulent flow, independently on the injection mechanism, recovers universal statistical properties, for scales small enough (and far from the boundaries). Indeed, experiments and numerical simulations give strong indications that Eulerian and Lagrangian isotropic/anisotropic small-scales velocity statistics are pretty independent of the *large-scale* forcing mechanisms [3–7]. Still, we lack a firm understanding for these evidences. From the theoretical point of view, precious hints arise from *linear* problems, like passive scalar or passively advected magnetic fields. For the class of Kraichnan models [8], anomalous scaling has been shown to be associated to statistically stationary solutions of the unforced equations for correlation functions [9]. Scaling exponents are consequently universal with respect of the injection mechanisms. Concerning non-linear problems, as the Navier-Stokes case, analytical results have been often pursued by means of the Renormalization Group (RG) [10, 11]. In the RG framework, turbulence is stirred at all scales by a self-similar Gaussian field, with zero mean and white-noise in time. The two-point correlation function in Fourier space is given by

$$\left\langle f_i(\boldsymbol{k},t) f_j(\boldsymbol{k}',t') \right\rangle \propto \quad (1)$$
$$D_0 k^{4-d}(k_0^2+k^2)^{-y/2} P_{ij}(\boldsymbol{k})\delta(\boldsymbol{k}+\boldsymbol{k}')\,\delta(t-t').$$

Here $1/k_0 \sim L$ is the largest length in the system (infrared cut-off), D_0 is the forcing intensity, $P_{ij}(\boldsymbol{k})$ is the projector assuring incompressibility and d is the spatial dimension (always assumed to be $d = 3$ hereafter). The influence of the stirring mechanism at small scales is governed by the value of the slope y. We go from a situation when the forcing has a strong input at all scales, $y \sim 0$ originally investigated in [10], to a quasi *large-scale* forcing when $y \to \infty$. Renormalization Group calculations, based on a y-expansion, predict a power-law energy spectrum $E(k) \sim k^{1-2y/3}$, in the domain $\eta \ll k^{-1} \ll L$, where η is the viscous scale of the system, and for $y \ll 1$. Notice that the Kolmogorov value, $E(k) \sim k^{-5/3}$, describing experimental turbulent flows stirred by a large-scale forcing, is obtained for $y = 4$, i.e. quite far from the perturbative region where the RG calculations are under control. The Kolmogorov spectrum can be obtained, however, by means of a simple dimensional analysis, still within the same framework [12]. Extension of the RG formalism to finite y values, up to $y = 4$, have been attempted with different kind of approximation [13, 14] altough in a range where convergence of the RG expansion is not granted anymore [15]. As for the numerical simulations, in [16] the problem has been investigated for various y values, and it has been shown that, for $y = 4$, results are in good agreement with the picture of large-scale forced turbulence, while for $y < 4$ the situation becomes less clear. However, the low numerical resolution used in [16] makes these results far from being conclusive.

Beside the issue connected to the RG approach, there exists a whole set of interesting questions concerning turbulent flows with a power-law forcing. To which extent small-scale fluctuations are sensitive to the injection mechanism? Does it exist a critical value y_c separating different regimes? Can we observe anomalous scaling in the forcing dominated case $y \sim 0$? Let us suppose, for example, that it exists a finite y_c, beyond which small-scale statistics is forcing independent: this would rule out any attempt to control intermittency analytically, by means of a perturbative approach which starts from forcing dominated turbulent solutions at $y \ll y_c$.

Hints on the problem can also come from the study of the one-dimensional Burgers equation in presence of a power law forcing. In [17], a numerical study was presented showing that there is a critical value of the forcing

slope, such that the velocity field passes from the usual bifractal statistics (observed in large-scale forced Burgers flows), to a statistics influenced from the forcing. Surprisingly, also in the forcing dominated regime, a non-trivial (multifractal?) scaling was observed in [17]. A rigorous understanding of the mechanism leading to this result is, however, still missing: we will come back to this point later on, after having presented our results.

In the sequel, we address the problem of the small-scale statistical properties of three dimensional turbulent flows at varying the parameter y. We will show that at crossing $y_c = 4$ there exist two regimes for the velocity field statistics with well defined and different scaling properties.

We solved the Navier-Stokes equations with a second-order hyperviscous dissipative term $\propto \nu\Delta^2$. Temporal integration has been carried over for about $20 \div 30$ large-eddy turnover times. We performed variuos experiments, at resolutions 128^3, 256^3 and 512^3, corresponding to a maximum Taylor's Reynolds number equal to $Re_\lambda = 220$ for the 512^3 run. As for the stirring force, we specialized in the two following cases, one for each regime: the first with $y = 3.5 < y_c$, the second with $y = 6 > y_c$. We also show some results obtained with an analityical large-scale forcing, i.e. the equivalent of $y \to \infty$. The range of the forcing, in Fourier space, extends down to the maximum resolved wavenumber. As we are always confined in a finite box, we neglect here possible subtle behaviours due to infrared divergences in the injection mechanism.

We start considering what happens to the system when the slope of the forcing is changed. It is instructive to consider the equation for the energy flux through the wavenumber k: $\Pi(k) \equiv -\pi k^2 \int \Im[\langle\, (\boldsymbol{k}\cdot\hat{\boldsymbol{v}}(\boldsymbol{q}))(\hat{\boldsymbol{v}}(\boldsymbol{k})\cdot\hat{\boldsymbol{v}}(\boldsymbol{p}))\,\rangle + \langle\, (\boldsymbol{k}\cdot\hat{\boldsymbol{v}}(\boldsymbol{p}))(\hat{\boldsymbol{v}}(\boldsymbol{k})\cdot\hat{\boldsymbol{v}}(\boldsymbol{q}))\,\rangle]\mathrm{d}\boldsymbol{p}\mathrm{d}\boldsymbol{q}$, where the three wavevectors satisfy $\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q} = 0$, and the symbol $\Im$ stands for the imaginary part. Such equation is equivalent to the Kármán-Howarth equation in physical space; it states that in a stationary, isotropic and homogeneous flow, the contribution to the energy flux $\Pi(k)$ due to the non-linear terms balances the total energy input from the injection mechanism (see [1, 18]):

$$\Pi(k) \sim \int_{k_0<|\boldsymbol{k}|<k} \Re(\langle \boldsymbol{f}(\boldsymbol{k})\boldsymbol{v}(-\boldsymbol{k})\rangle)\mathrm{d}\boldsymbol{k}, \qquad (2)$$

where $k_0 \sim 1/L$, the symbol $\Re$ stands for the real part and where we have neglected dissipative effects. For the special class of forcings (1), the *rhs* of (2) can be further simplified to: $\Pi(k) \sim \int_{k_0<|\boldsymbol{k}|<k} \left\langle |\boldsymbol{f}(\boldsymbol{k})|^2 \right\rangle \mathrm{d}\boldsymbol{k}$. From (1) the forcing spectrum is $E_f(k) = \left\langle |\boldsymbol{f}(\boldsymbol{k},t)|^2 \right\rangle \sim k^{3-y}$. It follows that for $y \geq 4$, the energy flux is constant in Fourier space for $kL \gg 1$ (up to logarithmic corrections for $y = 4$). In other words, the energy injection is dominated by the small wave-number region in the integral (2). In this case we expect to be very close to the typical experimental situation of turbulence with a large-scale, analytical forcing: energy is transferred down-scale *via* an intermittent cascade. Coherently, the third order longitudinal structure function, $S^{(3)}(r) \equiv \left\langle [(\boldsymbol{v}(\boldsymbol{x}+\mathbf{r}) - \boldsymbol{v}(\boldsymbol{x}))\cdot\hat{\boldsymbol{r}}]^3 \right\rangle$, follows a linear behaviour in r as predicted by the 4/5 law [1]. For $y < 4$, the energy flux no longer saturates to a constant value as a function of k. The integral in (2) now becomes ultraviolet dominated. The direct input of energy from the forcing mechanism affects inertial range statistics in a self-similar way, down to the smallest scales where dissipative terms start to be important. In this situation, we get for the energy flux $\Pi(k) \sim k^{4-y}$, with a constant prefactor which depends on the ultraviolet cut-off. The corresponding scaling behaviour for the third order structure function is now given by $S^{(3)}(r) \sim r^{y-3}$.

What about higher order statistics? One is tempted to guess that for $y \geq 4$ the fluctuations induced by the injection mechanism are always sub-leading, anomalous scaling being the result of the cascade mechanism driven by the non-linear terms of the equations of motion. If so, for $y > 4$ we should fall in the same class of "universality" of turbulence generated with large-scale forcing, i.e. small-scale velocity fluctuations should be universal. Therefore, as far as $y > 4$, longitudinal structure functions should scale as:

$$S^n(r) \equiv \langle [(\boldsymbol{v}(\boldsymbol{x}+\mathbf{r}) - \boldsymbol{v}(\boldsymbol{x}))\cdot\hat{\boldsymbol{r}}]^n \rangle \sim r^{\zeta_\infty^{(n)}}. \qquad (3)$$

In (3), we have denoted with ζ_∞^n the scaling exponents measured with a smooth large-scale forcing.

On the other hand, for $y < 4$, energy is directly injected in the inertial range. Here, a dimensional matching with the forcing gives a scaling behaviour which is always leading with respect to what predicted in the $y > 4$ range (3). We expect now that anomalous scaling disappears, everything being dominated by the Gaussian energy input at all scales. By the simple dimensional argument connecting the scaling of structure functions to that of the external forcing, for the range $y < 4$ we have:

$$S^{(n)}(r) \sim r^{\zeta_y^{(n)}} \;\text{ with }\; \zeta_y^{(n)} = \frac{n}{3}(y-3). \qquad (4)$$

In Fig. 1 we show the sixth order structure function, $S^{(6)}(r)$ for the two cases $y = 3.5$ and $y = 6$, compensated with the dimensional prediction given by the matching with the forcing (4). As it is clear, only for $y = 3.5$ the statistics follows the forcing injection obtaining a nice compensation. On the other hand, for $y = 6$ the statistics is much closer to what usually measured with an analytical large-scale forcing. This is quantitatively confirmed by the inset picture, where we plot the logarithmic derivatives of $S^{(6)}(r)$ *vs.* $S^{(3)}(r)$, for the two cases $y = 3.5$ and $y = 6$, together with the results of the simulation with a large-scale, analytical forcing corresponding to $y \to \infty$. Here the local slopes for $y = 6$ and $y \to \infty$ fluctuate

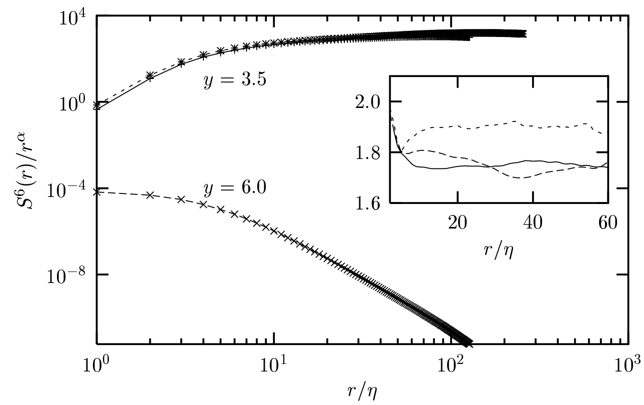

FIG. 1: Log-log plot of the compensated sixth-order structure function $S^{(6)}(r)/r^\alpha$. The two top curves are for $y = 3.5$ at the two resolutions 256^3 and 512^3: they are compensated with the dimensional scaling 4, i.e. with an exponent $\alpha = \zeta^{(6)}_{y=3.5} = 0.5$. The bottom curve refers to the case $y = 6$, at the resolutions 256^3, and is also compensated with the exponent for the scaling (4), $\alpha = \zeta^{(6)}_{y=6} = 6$. Clearly the matching with the dimensional exponent is not the correct one in the case $y = 6$. Inset: local slopes of the ESS curve for $S^{(6)}(r)$ *vs.* $S^{(3)}(r)$ at varying r. The top curve refers to the case $y = 3.5$, and the two bottom curves refer to the cases $y = 6$ and $y \to \infty$. The dimensional scaling would correspond to the value 2.

around the same value, compatible with those reported in literature [2, 4], while the local slope for $y = 3.5$ is different and tends to the expected dimensional value. Values of all scaling exponents obtained in the simulations are summarized in Table I. Let us notice that the measured exponents for the $y = 3.5$ case are very close to the non-intermittent, dimensional prediction. Only for high order moments (i.e. $n = 6$) there is a *small* deviation from the expected value $\zeta^{(6)}/\zeta^{(3)} = 2$.

To quantify the level of intermittency at changing the scale, we also plot the Probability Density Function (PDF) of velocity increments at different scales, normalized to have unit variance. Figure 2 shows the PDFs, in the case $y = 6$, for three different separations $r_1 = 34\eta$ and $r_2 = 74\eta$ in the inertial range, and $r_3 = 114\eta$ in the energy containing range. The three curves have larger tails than a Gaussian distribution and have an intermittent behaviour, i.e. they cannot be superposed. In Fig. 3 we show the PDFs, for the case $y = 3.5$, at the same separations (r_1, r_2, r_3). Now, the three curves are almost indistinguishable, and show a very good rescaling, a signature of the absence of intermittent effects. Only for negative increments, a very tiny discrepancy is measured. It is hard to say whether this is a robust effect or a spurious Reynolds dependent phenomenon. We will come back to this issue later in the conclusions. A dramatic difference at crossing the $y = y_c$ value is also observed in the energy dissipation statistics. For both cases $y = 3.5$ and $y = 6$, we measured the PDF's of the coarse-grained energy dissipation $\varepsilon_r(\boldsymbol{x}) = V_r^{-1} \int_{V_r(\boldsymbol{x})} \varepsilon(\boldsymbol{x}+\mathbf{r}) d\mathbf{r}$, where ε is the rate of dissipation for unit volume, and the volume $V_r(\boldsymbol{x})$ is centered at $\boldsymbol{x}$ and has characteristic length scale $r \ll L$. In Fig. 4 we compare the PDF's $\mathcal{P}(\varepsilon_r)$ at the scale $r = 8\eta$. Here the results are even more impressive as the shape change is particularly strong.

In this letter, we have presented clear evidences

n	1	2	4	5	6
y_d	0.333	0.666	1.33	1.66	2.00
$y = 3.5$	0.34(1)	0.67(1)	1.31(2)	1.62(2)	1.93(3)
$y = 6$	0.36(1)	0.69(1)	1.28(2)	1.53(3)	1.75(4)
$y \to \infty$	0.36(1)	0.69(1)	1.27(2)	1.52(3)	1.75(4)

TABLE I: Scaling exponents in ESS, of the curves $S^{(n)}(r)$ *vs.* $S^{(3)}(r)$, extracted from the following numerical simulations: $y = 3.5$, at resolution 256^3 and 512^3; $y = 6$, at resolution 256^3; $y \to \infty$, at resolution 512^3. The first raw describes the dimensional values: $\zeta^{(n)}/\zeta^{(3)} = n/3$

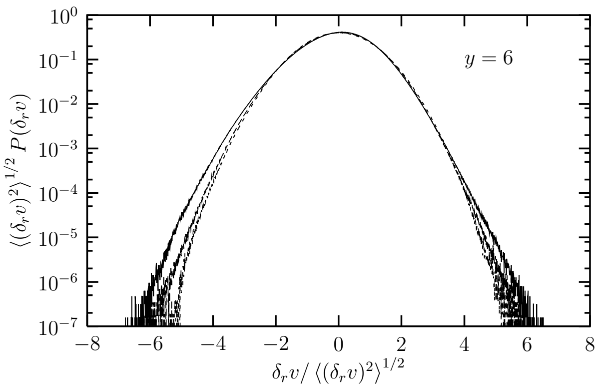

FIG. 2: PDF's of the velocity increments, for $y = 6.0$, for three separations $r_1 = 34\eta$ and $r_2 = 74\eta$ in the inertial range, and $r_3 = 114\eta$ in the energy containing range.

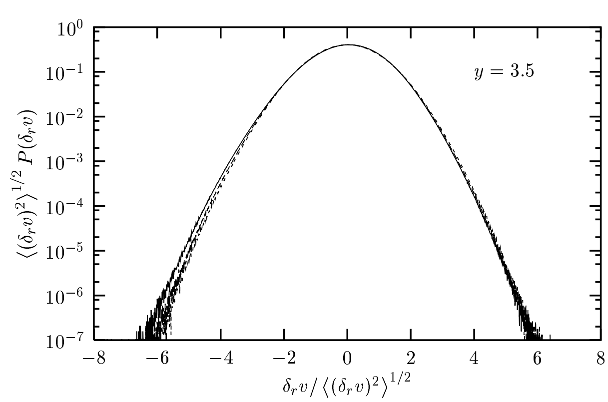

FIG. 3: PDF's of the velocity increments, for $y = 3.5$. Scales are the same of the previous picture: $r_1 = 34\eta$, $r_2 = 74\eta$, and $r_3 = 114\eta$.

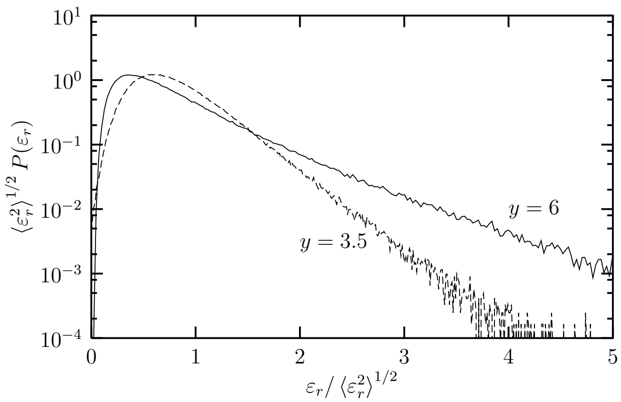

FIG. 4: PDF's of the coarse-grained energy dissipation $\mathcal{P}(\varepsilon_r)$, for both $y = 3.5$ and $y = 6.0$, at the scale $r = 8\eta$.

that turbulent small-scale fluctuations, in presence of a direct injection of energy at all scales, undergo a sharp transition for $y_c = 4$. The first regime, for $y > y_c$, is mainly forcing independent: small-scale fluctuations develop anomalous scaling in agreement with what observed in experiments and/or numerics obtained with a large-scale forcing. This is a stringent test of turbulence universality: even if directly affected by the injection of energy, small-scale fluctuations show a robust behaviour. Things change abruptly at the critical value of y_c, where the direct injection of energy becomes the dominant effect in the inertial range. In this second regime, corresponding to $y < y_c$, small-scale fluctuations get closer and closer to a Gaussian statistics and intermittency disappear [19]. Before concluding, we discuss two possible mechanisms which could partially disprove the last statement. First, even when $y < y_c$, we may imagine that the intermittent energy cascade dominating the statistics for $y > y_c$ might show up. For example, we may have that for high order moments the forcing dominated solutions become subleading with respect to those associated to the cascading mechanism. In such case, the loss of rescaling in the PDF's tail of Fig. 2, may be due to the survival of these rare anomalous fluctuations. Second, even more complex is the scenario proposed in [17], where the possibility to have a forcing dependent multiscaling statistics, when $y < y_c$, is conceived. This is not the case for the linear Kraichnan models [9], where forcing dependent solutions are always dimensionally scaling. The main difference is that, in the Navier-Stokes problem, the hierarchy of equations for correlation function is unclosed: one cannot solve it for a single order independently of all the others. In NS, being the low order moments always forcing dominated for $y < y_c$, one may observe some forcing dependency also on high order fluctuations, *via* their coupling with low order correlation functions. This may be a possible explanation of the multifractal behaviour observed in Burgers equation [17].

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[1] U. Frisch, *Turbulence: The Legacy of A. N. Kolmogorov*, Cambridge University Press, Cambridge (1995).
[2] K. R. Sreenivasan and R. A. Antonia, Annu. Rev. Fluid Mech. **29**, 435 (1997).
[3] X. Shen and Z. Warhaft, Phys. Fluids **14**, 370 (2002).
[4] T. Gotoh, D. Fukayama, and T. Nakano, Phys. Fluids, **14**, 1065 (2002).
[5] A. La Porta, G. A. Voth, A. M. Crawford, J. Alexander, and E. Bodenschatz, Nature **409**, 1017 (2001).
[6] L. Biferale, G. Boffetta, A. Celani, A. Lanotte, F. Toschi, & M. Vergassola, Phys. Fluids, **15**, 2105 (2003).
[7] L. Biferale, E. Calzavarini, F. Toschi, and R. Tripiccione, Europhys. Lett., in press (2003).
[8] R. H. Kraichnan, Phys. Rev. Lett. **72**, 1016 (1994).
[9] G. Falkovich, K. Gawędzki, and M. Vergassola, Rev. Mod. Phys. **73**, 913 (2001).
[10] D. Forster, D. R. Nelson, and M. J. Stephen, Phys. Rev. A **16**, number 2, 732 (1977).
[11] U. Frisch and J. D. Fournier, Phys. Rev. A **17**, 747 (1978).
[12] C. De Dominicis and P. C. Martin, Phys. Rev. A **19**, number 1, 419 (1979).
[13] V. Yakhot and S. A. Orszag, Phys. Rev. Letters **57**, 1722 (1986).
[14] L. Ts. Adzhemyan, N.V. Antonov, M. V. Kompaniets, and A. N. Vasil'ev, Intern. Journ. Modern Physics B, **17**, No. 10, 2137 (2003).
[15] G. Eyink, Phys. Fluids **6**, 3063 (1994).
[16] A. Sain, Manu, and R. Pandit, Phys. Rev. Letters **81**, 4377 (1998).
[17] F. Hayot and C. Jayaprakash, Phys. Rev.E **56**, 4259 (1997); same authors, Phys. Rev.E **56**, 227 (1997).
[18] A.S. Monin and A.M. Yaglom *Statistical Fluid Mechanics*, MIT Press, Cambridge MA, (1975).
[19] In the present work we refrained from investigating the case with $y < 3$. In this range velocity structure functions may become UV divergent and important effects due to the matching with the dissipative region may show up.